

A SHORT NOTE ON THE MULTIPLIER IDEALS OF MONOMIAL SPACE CURVES

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ABSTRACT. Thompson (2014) exhibits a formula for the multiplier ideal with multiplier λ of a monomial curve C with ideal I as an intersection of a term coming from the I -adic valuation, the multiplier ideal of the term ideal of I , and terms coming from certain specified auxiliary valuations. This short note shows it suffices to consider at most two auxiliary valuations. This improvement is achieved through a more intrinsic approach, reduction to the toric case.

1. INTRODUCTION

Let (Y, Δ) be a pair, consisting of a normal variety Y over an algebraically closed field of characteristic zero and a \mathbb{Q} -divisor Δ such that $K_Y + \Delta$ is \mathbb{Q} -Cartier. Let $\pi : X \rightarrow Y$ be a log resolution of the ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$ that is also a log resolution of the pair (Y, Δ) . That is, π is a proper birational morphism such that X is smooth, the union of the exceptional set of π and $\pi^{-1}(\Delta)$ is a divisor with simple normal crossing support, and $\mathcal{I} \cdot \mathcal{O}_X = \mathcal{O}_X(-F)$ is also a divisor with simple normal crossing support. In this setting, we define the multiplier ideal of \mathcal{I}^λ on the pair (Y, Δ) to be

$$\mathfrak{J}((Y, \Delta), \mathcal{I}^\lambda) = \pi_* \mathcal{O}_X(K_X - \lfloor \pi^*(K_Y + \Delta) + \lambda F \rfloor).$$

This ideal sheaf on Y does not depend upon the choice of log resolution.

In recent years, researchers have begun to study which divisors on a log resolution contribute jumping numbers. See Alberich-Carramiñana, Álvarez Montaner and Dachs-Cadefau [1], Galindo and Monserrat [6], Hyry and Järvilehto [10], Naie [11], Naie [12], Smith and Thompson [14], and Tucker [18]. This paper refines the result of Thompson [17]

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by finding a smaller set of divisors that contains all the divisors that contribute jumping numbers for a monomial space curve.

Section 2 of this paper recalls a strengthening of the notion of an embedded resolution of singularities known as a factorizing resolution and uses it to provide a proposition (Proposition 3 on the next page) about the structure of multiplier ideals.

Section 3 on the facing page of this paper recalls the Howald-Blickle Theorem (Proposition 4 on page 4) that provides a formula for the multiplier ideals of a monomial ideal on a normal affine toric variety, provides a reinterpretation (Proposition 6 on page 4) of that theorem, and provides a formula (Proposition 7 on page 4) for the multiplier ideals of a principal binomial ideal.

Section 4 on page 5 applies the ideas of the previous sections to refine the result of Thompson [17].

2. USING FACTORIZING RESOLUTIONS TO COMPUTE MULTIPLIER IDEALS

Definition 1. Let Z be a generically smooth subscheme of any variety Y . A *factorizing resolution* of Z is an embedded resolution $\pi : X \rightarrow Y$ of Z such that

$$\mathcal{I}_Z \cdot \mathcal{O}_X = \mathcal{I}_{\tilde{Z}} \cdot \mathcal{L}$$

where \tilde{Z} is the strict transform of Z , \mathcal{L} is an invertible sheaf, and the support of $\mathcal{I}_Z \cdot \mathcal{O}_X$ is a simple normal crossings variety.

Recall that π is an embedded resolution of Z if it is proper birational morphism $\pi : X \rightarrow Y$ such that: X is smooth and π is an isomorphism over the generic points of the components of Z , the exceptional locus $\text{exc}(\pi)$ of π is a divisor with simple normal crossing support, and the strict transform \tilde{Z} is smooth and transverse to $\text{exc}(\pi)$. For an embedded resolution, we always have $\mathcal{I}_Z \cdot \mathcal{O}_X = \mathcal{I}_{\tilde{Z}} \cap \mathcal{L}$ for some invertible sheaf \mathcal{L} . Here we require the intersection to be a product. Typically, this is achieved by blowing up embedded components of $\mathcal{I}_Z \cdot \mathcal{O}_X$. Here is a theorem on the existence of factorizing resolutions.

Proposition 2. (*Theorem 1.2 of Bravo [3], Section 3 of Eisenstein [5]*) *Let Z be a generically smooth subscheme of any variety Y over an algebraically closed field of characteristic zero such that there exists a birational morphism $\mu_1 : Y' \rightarrow Y$ from a smooth variety Y' that is an isomorphism over the generic points of the components of Z . If D is a divisor on Y' with simple normal crossing support such that no component of the strict transform of Z is contained in D , then there exists a factorizing resolution $\pi : X \rightarrow Y$ of Z that factors through μ_1 ,*

$\pi = \mu_2 \circ \mu_1$, such that $\tilde{Z} \cup \text{exc}(\pi) \cup \mu_2^{-1}(D)$ has simple normal crossing support.

Notice that if $\pi : X \rightarrow Y$ is such a factoring resolution of Z , then the blowup of \tilde{Z} is a log resolution of Z and that the exceptional locus of this blowup consists of a collection of prime divisors in one-to-one correspondence with the components of Z with codimension at least two.

Proposition 3. *Let Z_1, \dots, Z_r be the components of Z and suppose e_i is the codimension of Z_i for all i . Fix a factorizing resolution $\pi : X \rightarrow Y$ of Z that is also a log resolution of the pair (Y, Δ) and let $\mathfrak{b} = \pi_*(\mathcal{L})$ where $\mathcal{I}_Z \cdot \mathcal{O}_X = \mathcal{I}_{\tilde{Z}} \cdot \mathcal{L}$ as above. Then,*

$$\mathfrak{J}((Y, \Delta), \mathcal{I}_Z^\lambda) = \mathfrak{J}((Y, \Delta), \mathfrak{b}^\lambda) \cap \bigcap_{i=1}^r \mathcal{I}_{Z_i}^{(\lfloor \lambda+1-e_i \rfloor)}$$

Proof. Since $\mathcal{I}_Z \subseteq \mathfrak{b}$, it is clear that $\mathfrak{J}((Y, \Delta), \mathcal{I}_Z^\lambda) \subseteq \mathfrak{J}((Y, \Delta), \mathfrak{b}^\lambda)$. Let us now show $\mathfrak{J}((Y, \Delta), \mathcal{I}_Z^\lambda) \subseteq \mathcal{I}_{Z_i}^{(\lfloor \lambda+1-e_i \rfloor)}$ for each i . Since Z is generically smooth, the \mathcal{I}_{Z_i} are prime and the $\mathcal{I}_{Z_i}^{(\lfloor \lambda+1-e_i \rfloor)}$ are primary. Since $\mathcal{I}_{Z_i}^{(\lfloor \lambda+1-e_i \rfloor)}$ is primary, it suffices to check the inclusion generically along the corresponding component (that is, after localizing at \mathcal{I}_{Z_i}). Because Z is generically reduced, $\mathcal{I}_Z \cdot \mathcal{O}_{Z_i} = \mathcal{I}_{Z_i} \cdot \mathcal{O}_{Z_i}$. It is simple calculation based on the fact that one can resolve \mathcal{O}_{Z_i} generically by blowing up \mathcal{O}_{Z_i} , and the relative canonical divisor for this blowup is $(e_i - 1)E_i$, where E_i is the resulting exceptional divisor.

On the other hand, the extension of the contraction of an ideal is contained in the ideal. So, $\mathfrak{b} \cdot \mathcal{O}_X \subseteq \mathcal{L}$. Thus,

$$\mathfrak{b} \cdot \mathcal{O}_X \cap \bigcap_{i=1}^r \mathcal{I}_{\tilde{Z}_i} \subseteq \mathcal{L} \cap \bigcap_{i=1}^r \mathcal{I}_{\tilde{Z}_i} = \mathcal{I}_Z \cdot \mathcal{O}_X.$$

Therefore, we see

$$\mathfrak{J}((Y, \Delta), \mathfrak{b}^\lambda) \cap \bigcap_{i=1}^r \mathcal{I}_{Z_i}^{(\lfloor \lambda+1-e_i \rfloor)} \subseteq \mathfrak{J}((Y, \Delta), \mathcal{I}_Z^\lambda)$$

using any log resolution of \mathfrak{b} , Z and (Y, Δ) that factors through π . \square

3. EXPLOITING THE TORIC CASE

This paragraph is, essentially, a direct quote of Blickle [4]. Let (Y, Δ) be a pair such that Y is a normal (affine) toric variety (say $Y = \text{Spec } R$ for some normal semigroup ring $R \subseteq \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$) and Δ is a torus invariant \mathbb{Q} -divisor. Since $K_Y + \Delta$ is \mathbb{Q} -Cartier and torus invariant,

there is a monomial $\mathbf{x}^{\mathbf{u}}$ such that $\operatorname{div} \mathbf{x}^{\mathbf{u}} = r(K_Y + \Delta)$ for some integer r . Set $\mathbf{w} = \mathbf{u}/r$. Blickle's version of Howald's [8] formula is the following.

Proposition 4. (*Theorem 1 of Blickle [4]*) *Let \mathfrak{a} be a monomial ideal on Y . Then, if $\operatorname{Newt}(\mathfrak{a})$ is the Newton Polyhedron of \mathfrak{a} ,*

$$\mathfrak{J}((Y, \Delta), \mathfrak{a}^\lambda) = \langle \mathbf{x}^{\mathbf{v}} \in R \mid \mathbf{v} + \mathbf{w} \in \text{interior of } \lambda \operatorname{Newt}(\mathfrak{a}) \rangle$$

for all $\lambda > 0$.

This means that the multiplier ideals of a monomial ideal on a toric variety are contributed by (divisors supported on unions of) the Rees divisors of the ideal. (See Thompson [16] for a quick overview of the relationship between toric blowups and Newton polyhedra.) Other divisors that may appear on a log resolution do not contribute.

Definition 5. We will say a sheaf \mathcal{F} (respectively a Weil divisor D) on X is *locally monomial* if X can be covered by open subschemes U such that each U is isomorphic to an open subscheme of a normal toric variety in such a way that $\mathcal{F}(U)$ is identified with a torus invariant sheaf (resp. a torus invariant divisor).

Proposition 6. *Let (Y, Δ) be a pair, consisting of a normal variety Y over an algebraically closed field of characteristic zero and a \mathbb{Q} -divisor Δ such that $K_Y + \Delta$ is \mathbb{Q} -Cartier. If $\pi : X \rightarrow Y$ is a proper birational morphism such that $\pi^{-1}(\Delta)$ and $\mathcal{I} \cdot \mathcal{O}_X$ are locally monomial, then the multiplier ideals of \mathcal{I} are contributed by the Rees divisors of $\mathcal{I} \cdot \mathcal{O}_X$.*

Proof. Since the question is local on X , it suffices to consider the case where X is a normal affine toric variety and $\mathfrak{a} = \mathcal{I} \cdot \mathcal{O}_X$ is a monomial ideal. Let $\mu : X' \rightarrow X$ be a toric log resolution of the ideal \mathfrak{a} that is also a log resolution of the pair $(X, \pi^{-1}(\Delta))$. Evidently, μ factors through the blowup of \mathfrak{a} and, as in the toric case, orders of vanishing on any exceptional divisor of $\mu \circ \pi$ are determined by those on the blowup of \mathfrak{a} . This is just the fact that when one represents a polyhedron as an intersection of half-spaces it suffices to consider only the facet-defining half-spaces. \square

Consider the case of any principal binomial ideal $I = \langle \mathbf{x}^{\mathbf{v}_1} - \mathbf{x}^{\mathbf{v}_2} \rangle \subseteq \mathbb{k}[x_1, \dots, x_n]$.

Proposition 7. *If $I = \langle \mathbf{x}^{\mathbf{v}_1} - \mathbf{x}^{\mathbf{v}_2} \rangle$ and $\mathfrak{t} = \langle \mathbf{x}^{\mathbf{v}_1}, \mathbf{x}^{\mathbf{v}_2} \rangle$ is the term ideal of I , then*

$$\mathfrak{J}(\mathbb{A}^n, I^\lambda) = \mathfrak{J}(\mathbb{A}^n, \mathfrak{t}^\lambda) \cap I^{(\lfloor \lambda \rfloor)}.$$

Proof. Let \mathbf{v} be a primitive lattice vector such that $r\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ for some positive integer r . Cover the normalized blowup of \mathfrak{t} with affine

open toric varieties U_1, \dots, U_s . Now, consider the covering consisting of the open sets of the form $\text{Spec } \mathbb{k}[x_1, \dots, x_n, \mathbf{x}^{\pm \mathbf{v}}] \setminus V(f)$ where $f = \frac{\mathbf{x}^{r\mathbf{v}} - 1}{\mathbf{x}^{\mathbf{v}} - \zeta}$ for an r th root of unity ζ and the open subsets obtained by removing the closure of $V(\mathbf{x}^{r\mathbf{v}} - 1) \subseteq \text{Spec } \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ from each U_i .

Note that since \mathbf{v} is primitive, $\mathbb{Z}\mathbf{v}$ splits from \mathbb{Z}^n . So, $\mathbb{N}^n + \mathbb{Z}\mathbf{v} \cong S \times \mathbb{Z}$ where S is the image of \mathbb{N}^n in $\mathbb{Z}^n / \mathbb{Z}\mathbf{v}$. Each open set of the form $\text{Spec } \mathbb{k}[x_1, \dots, x_n, \mathbf{x}^{\pm \mathbf{v}}] \setminus V(f)$ is of the form $\text{Spec } \mathbb{k}[S][t]_{t+1}$ where $t = \mathbf{x}^{\mathbf{v}} - 1$.

And, on each open set of the form $U_i \setminus V(\mathbf{x}^{r\mathbf{v}} - 1)$, $I \cdot \mathcal{O}_X = \mathfrak{t} \cdot \mathcal{O}_X$ is monomial already. Thus, Proposition 6 on the preceding page applies. Moreover, it is clear that the components of the closure of $V(\mathbf{x}^{r\mathbf{v}} - 1) \subseteq \text{Spec } \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ are smooth and meet the boundary transversely.

So, a toric desingularization of the blowup of $\mathfrak{t} \cdot \mathcal{O}_X$ is a factorizing resolution of I . Now, apply Proposition 3 on page 3. \square

This result is not new. Principal binomial ideals are nondegenerate. For an alternate proof, see Howald [9].

4. APPLICATION TO THE MONOMIAL SPACE CURVE CASE

Using the previous ideas, one can refine the result of Thompson [17]. The case where the monomial space curve is contained in a smooth toric surface follows from the principal toric case by using adjunction and inversion of adjunction.

Let \mathbb{k} be a field of characteristic zero, let $C = \{(t^{n_1}, t^{n_2}, t^{n_3})\} \subset \mathbb{A}_{\mathbb{k}}^3$ be a monomial space curve not contained in a smooth toric surface. Assume $\mathbf{n} = [n_1 \ n_2 \ n_3] \in \mathbb{Z}_{>0}^3$ is a primitive vector, let $\text{ord}_{\mathbf{n}}$ be the monomial valuation given by the standard pairing, $\mathbf{x}^{\mathbf{m}} \mapsto \langle \mathbf{n}, \mathbf{m} \rangle$, and let $I \subset \mathbb{k}[x_1, x_2, x_3]$ be the ideal of C . We may assume there exist irreducible binomials $f_1 = \mathbf{x}^{\mathbf{m}_1^+} - \mathbf{x}^{\mathbf{m}_1^-}$, $f_2 = \mathbf{x}^{\mathbf{m}_2^+} - \mathbf{x}^{\mathbf{m}_2^-}$, and $f_3 = \mathbf{x}^{\mathbf{m}_3^+} - \mathbf{x}^{\mathbf{m}_3^-}$ such that $\{f_1, f_2, f_3\}$ or $\{f_1, f_2\}$ is a minimal generating set for I . Let $\mathfrak{t} = (\mathbf{x}^{\mathbf{m}_1^+}, \mathbf{x}^{\mathbf{m}_1^-}, \mathbf{x}^{\mathbf{m}_2^+}, \mathbf{x}^{\mathbf{m}_2^-}, \mathbf{x}^{\mathbf{m}_3^+}, \mathbf{x}^{\mathbf{m}_3^-})$ be the term ideal of I . Let $d_i = \text{ord}_{\mathbf{n}}(f_i)$ for $i = 1, 2, 3$. Order the generators so that $d_1 < d_2 < d_3$ and order the n_i so that $n_i | d_i$ for $i = 1, 2$ (and $n_3 | d_3$ when f_3 is a minimal generator). See Section 3 of Shibuta and Takagi [13] for a more detailed setup. Let $\mathbf{m}_1 = \mathbf{m}_1^+ - \mathbf{m}_1^-$ and let $\mathbf{q} = [q_1 \ q_2 \ 0] \in \mathbb{N}^3$ be the primitive vector such that $\langle \mathbf{q}, \mathbf{m}_1 \rangle = 0$. And, let $e_i = \text{ord}_{\mathbf{q}}(f_i)$ for $i = 1, 2, 3$.

Proposition 8. *Let $\mathbf{a}_1 = (\mathbf{x}^{\mathbf{m}_1^+}, \mathbf{x}^{\mathbf{m}_1^-})$, let $\mathbf{a}_2 = (x_1^{n_2 n_3}, x_2^{n_1 n_3}, x_3^{n_1 n_2})$, and let the toric variety $X = X_{\Sigma}$ be the normalized blowup of $\mathbf{a}_1 \mathbf{a}_2$. Then the ideal sheaf $I \cdot \mathcal{O}_X$ is locally monomial.*

Proof. The blowup of \mathbf{a}_1 is the partial desingularization of the toric surface $V(f_1)$ identified in González Pérez and Teissier [7], and the normalized blowup of \mathbf{a}_2 is the partial desingularization of C . The fan Σ_1 of the blowup of \mathbf{a}_1 has two maximal cones $\{\mathbf{v} \in \mathbb{R}_{\geq 0}^3 \mid \langle \mathbf{v}, \mathbf{m}_1 \rangle \leq 0\}$ and $\{\mathbf{v} \in \mathbb{R}_{\geq 0}^3 \mid \langle \mathbf{v}, \mathbf{m}_1 \rangle \geq 0\}$. The normalized blowup of \mathbf{a}_2 is stellar subdivision along the ray $\rho = \mathbb{R}_{\geq 0}\mathbf{n}$. Note that \mathbf{n} is in the intersection of the two maximal cones of Σ_1 . So, the two operations on fans, stellar subdivision along \mathbf{n} and cutting with the plane $\{v \in \mathbb{R}_{\geq 0}^3 \mid \langle \mathbf{v}, \mathbf{m}_1 \rangle = 0\}$ commute. And, Σ is the stellar subdivision along ρ of Σ_1 . (Any toric desingularization of X provides a common embedded desingularization of C and the surface $V(f_1)$.)

First, consider the affine open U_ρ of X and fix an element of the affine semigroup $\mathbf{m}_\rho \in \mathbf{S}_\rho$ such that $\langle \mathbf{m}_\rho, \mathbf{n} \rangle = 1$. I claim, $\{\mathbf{m}_1, \mathbf{m}_2\}$ is a basis of the kernel of the matrix $\begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}$, $\mathbf{S}_\sigma = \mathbb{N}^3 + \mathbb{Z}\mathbf{m}_1 + \mathbb{Z}\mathbf{m}_2$, $f_i = (\mathbf{x}^{\mathbf{m}_\rho})^{d_i}(\mathbf{x}^{\mathbf{m}_i} - 1)$ for each $i = 1, 2, 3$, and $f_3 \in (f_1, f_2)\mathbb{k}[\mathbf{S}_\sigma]$. So,

$$\begin{aligned} I \cdot \mathcal{O}_{U_\rho} &= ((\mathbf{x}^{\mathbf{m}_\rho})^{d_1}(\mathbf{x}^{\mathbf{m}_1} - 1), (\mathbf{x}^{\mathbf{m}_\rho})^{d_2}(\mathbf{x}^{\mathbf{m}_2} - 1)) \\ &= (\mathbf{x}^{d_1\mathbf{m}_\rho}) \cap (\mathbf{x}^{\mathbf{m}_1} - 1, \mathbf{x}^{d_2\mathbf{m}_\rho}) \cap (\mathbf{x}^{\mathbf{m}_1} - 1, \mathbf{x}^{\mathbf{m}_2} - 1) \end{aligned}$$

is monomial in $\mathbf{x}^{\mathbf{m}_\rho}$, $\mathbf{x}^{\mathbf{m}_1} - 1$, and $\mathbf{x}^{\mathbf{m}_2} - 1$. Since $d_1 < d_2$, there is an embedded component supported on the intersection of the strict transform of the surface $V(f_1)$ and the divisor D_ρ . Away from this embedded component, $I \cdot \mathcal{O}_X = \mathfrak{t} \cdot \mathcal{O}_X$. Thus, it suffices to check the closed points where the curve $\overline{V(\mathbf{x}^{\mathbf{m}_1} - 1, \mathbf{x}^{\mathbf{m}_\rho})}$ meets $X \setminus U_\rho$.

Let $p \in X$ be one of these two points, and let σ be the smallest cone of Σ such that $p \in U_\sigma$. Evidently, $\rho \subsetneq \sigma$ since $p \notin U_\rho$. After possibly replacing \mathbf{m}_1 with $-\mathbf{m}_1$, we may assume $\mathbf{x}^{\mathbf{m}_1} - 1 \in \mathbf{m}_{X,p}$. Since $\mathbf{x}^{\mathbf{m}_1} - 1 \in \mathbf{m}_{X,p}$, p is not a torus-fixed point and σ is two-dimensional.

Let $\sigma = \mathbb{R}_{\geq 0}^2 \begin{bmatrix} n_1 & n_2 & n_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$. Note that \mathbf{m}_1 is a basis for the kernel of $\begin{bmatrix} n_1 & n_2 & n_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$, and $\langle \mathbf{n}, \mathbf{m}_i \rangle = 0$ for $i = 1, 2, 3$. After possibly replacing \mathbf{m}_2 with $-\mathbf{m}_2$ and \mathbf{m}_3 with $-\mathbf{m}_3$, we may assume $\mathbf{m}_2, \mathbf{m}_3 \in \mathbf{S}_\sigma = \mathbb{N}^3 + \mathbb{Z}\mathbf{m}_1$. As in Proposition 7 on page 4, the affine semigroup is a product $\mathbf{S}_\sigma \cong \mathbf{S} \times \mathbb{Z}\mathbf{m}_1$ where \mathbf{S} is the image of \mathbb{N}^3 in the quotient $\mathbb{Z}^3/\mathbb{Z}\mathbf{m}_1$. Thus, $\mathbb{k}[\mathbf{S}_\sigma] \cong \mathbb{k}[\mathbf{S}][t_3]_{t_3+1}$ where $t_3 = \mathbf{x}^{\mathbf{m}_1} - 1$. \square

Recall $\mathbf{q} = [q_1 \ q_2 \ 0] \in \mathbb{N}^3$ is the primitive vector such that $\langle \mathbf{q}, \mathbf{m}_1 \rangle = 0$ and $e_i = \text{ord}_{\mathbf{q}}(f_i)$ for $i = 1, 2, 3$. Here is the improvement to the main theorem of Thompson [17].

Proposition 9. (i) If I is a complete intersection or if $e_2(d_3 - d_1) \leq e_1(d_3 - d_2)$, then

$$\mathfrak{J}(I^\lambda) = I^{(\lfloor \lambda - 1 \rfloor)} \cap \mathfrak{J}(\mathfrak{t}^\lambda) \cap (f \mid \nu_1(f) \geq \lfloor a_1 \lambda - k_1 \rfloor)$$

where ν_1 is the valuation given by the generating sequence $x_i \mapsto n_i$ for $i = 1, 2, 3$, $f_1 \mapsto d_2$. Thus, $a_1 = \nu_1(I) = d_2$ and $k_1 = \nu_1(J_{R_{\nu_1}/\mathbb{k}[x]}) = n_1 + n_2 + n_3 + d_2 - d_1$ where $J_{R_{\nu_1}/\mathbb{k}[x]}$ is the Jacobian of the discrete valuation ring R_{ν_1} of ν_1 over $\mathbb{k}[x]$.

(ii) Otherwise,

$$\mathfrak{J}(I^\lambda) = I^{(\lfloor \lambda - 1 \rfloor)} \cap \mathfrak{J}(\mathfrak{t}^\lambda) \bigcap_{i=1,2} (f \mid \nu_i(f) \geq \lfloor a_i \lambda - k_i \rfloor)$$

where ν_1 is as before and ν_2 is given by the generating sequence $x_1 \mapsto e_2 n_1 + (d_3 - d_2)q_1$, $x_2 \mapsto e_2 n_2 + (d_3 - d_2)q_2$, $x_3 \mapsto e_2 n_3$, $f_1 \mapsto e_2 d_3$. Thus, $a_2 = \nu_2(I) = e_2 d_3$, and $k_2 = \nu_2(J_{R_{\nu_2}/\mathbb{k}[x]}) = e_2(n_1 + n_2 + n_3) + (d_3 - d_1)(q_1 + q_2) + e_2(d_3 - d_1) - e_1(d_3 - d_2)$ where $J_{R_{\nu_2}/\mathbb{k}[x]}$ is the Jacobian of the discrete valuation ring R_{ν_2} of ν_2 over $\mathbb{k}[x]$.

Proof. Apply Proposition 8 on page 5 and the convex geometry computation in the appendix. \square

Corollary 10. (i) If I is a complete intersection or if $e_2(d_3 - d_1) \leq e_1(d_3 - d_2)$, then the log canonical threshold of I (at the origin) is

$$\text{lct}_0(I) = \min \left(\text{lct}_0(\mathfrak{t}), \frac{k_1 + 1}{a_1} \right).$$

(ii) Otherwise,

$$\text{lct}_0(I) = \min \left(\text{lct}_0(\mathfrak{t}), \frac{k_1 + 1}{a_1}, \frac{k_2 + 1}{a_2} \right).$$

Note that when $e_2(d_3 - d_1) = e_1(d_3 - d_2)$, ν_2 is monomial in the \mathbf{x} -variables and both formulas apply. In Example 5.3 of Blanco and Encinas [2], $e_2(d_3 - d_1) = e_1(d_3 - d_2)$, ν_2 . I do not know an example where $e_2(d_3 - d_1) > e_1(d_3 - d_2)$. A Macaulay2 package that implements this calculation, as presented in Thompson [17], is described in Teitler [15].

APPENDIX

Recall that $d_i = \text{ord}_{\mathbf{n}}(f_i)$ and let $\mathbf{u}_i = \begin{bmatrix} d_i \\ e_i \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & n_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \mathbf{m}_i^-$ for $i = 1, 2, 3$. In the local monomial coordinates, we find the Rees valuations from the facets of the Newton polyhedron. It suffices to

consider the ideal $(\mathbf{t}^{u_1}t_3, \mathbf{t}^{u_2}, \mathbf{t}^{u_3}) \subset \mathbb{k}[\mathbf{S}]ty_3]$ (see the conclusion of the proof of Proposition 9 on page 6). We know $e_1 > e_2$ by examining Section 3 of Shibuta and Takagi [13]. And, $r_2 = 0$ or $r_3 = 0$.

If $r_2 = 0$, then $e_2 = 0$, $\mathbf{t}^{u_3} \in (\mathbf{t}^{u_2})$, and the facets of the Newton polyhedron $\text{Newt}(\mathbf{t}^{u_1}t_3, \mathbf{t}^{u_2})$ are orthogonal to the rows of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ e_1 & d_2 - d_1 & 0 \\ 1 & 0 & d_2 - d_1 \end{bmatrix}$$

This includes the complete intersection case. Note that the only two rows of our matrix that have a nonzero last entry are $[0 \ 0 \ 1]$ and $[1 \ 0 \ d_2 - d_1]$. The other vectors correspond to valuations that are monomial in the original \mathbf{x} -variables. Our ideal $(\mathbf{t}^{u_1}t_3, \mathbf{t}^{u_2}, \mathbf{t}^{u_3})$ has order zero on the valuation corresponding to $[0 \ 0 \ 1]$. And, the row $[1 \ 0 \ d_2 - d_1]$ corresponds to ν_1 .

If $\mathbf{t}^{u_2} \notin \overline{(\mathbf{t}^{u_1}, \mathbf{t}^{u_3})}$ and $e_2 \neq 0$, then the facets of the Newton polyhedron $\text{Newt}(\mathbf{t}^{u_1}t_3, \mathbf{t}^{u_2}, \mathbf{t}^{u_3})$ are orthogonal to the rows of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ e_1 - e_2 & d_2 - d_1 & 0 \\ 1 & 0 & d_2 - d_1 \\ e_2 & d_3 - d_2 & 0 \end{bmatrix}$$

and these rows all have nonnegative integer entries. In terms of the parameters introduced in Section 3 of Shibuta and Takagi [13], $\alpha \leq \gamma$ in this case.

If $\mathbf{t}^{u_2} \in \overline{(\mathbf{t}^{u_1}, \mathbf{t}^{u_3})}$ and $e_2 \neq 0$, then $r_3 = 0$ and the facets of the Newton polyhedron $\text{Newt}(\mathbf{t}^{u_1}t_3, \mathbf{t}^{u_2}, \mathbf{t}^{u_3})$ are orthogonal to the rows of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ e_1 - e_3 & d_3 - d_1 & 0 \\ 1 & 0 & d_2 - d_1 \\ e_2 & d_3 - d_2 & e_2(d_3 - d_1) - e_1(d_3 - d_2) \end{bmatrix}$$

and these rows all have nonnegative integer entries. Note that the only three rows that have a nonzero last entry are $[0 \ 0 \ 1]$, $[1 \ 0 \ d_2 - d_1]$,

and

$$\begin{bmatrix} e_2 & d_3 - d_2 & e_2(d_3 - d_1) - e_1(d_3 - d_2) \end{bmatrix}$$

corresponding to the only bounded facet of $\text{Newt}(\mathbf{t}^{\mathbf{u}_1}t_3, \mathbf{t}^{\mathbf{u}_2}, \mathbf{t}^{\mathbf{u}_3})$. The other vectors correspond to valuations that are monomial in the original \mathbf{x} -variables. And, the bounded facet corresponds to ν_2 . For ν_2 , the orders of vanishing of the x -variables are given by the entries of

$$\begin{aligned} \begin{bmatrix} e_2 & d_3 - d_2 & e_2(d_3 - d_1) - e_1(d_3 - d_2) \end{bmatrix} & \begin{bmatrix} n_1 & n_2 & n_3 \\ q_1 & q_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ & = \begin{bmatrix} e_2n_1 + (d_3 - d_2)q_1 & e_2n_2 + (d_3 - d_2)q_2 & e_2n_3 \end{bmatrix} \end{aligned}$$

and $\nu_2(f_i) = e_2d_3$ for all $i = 1, 2, 3$

$$\begin{aligned} \begin{bmatrix} e_2 & d_3 - d_2 & e_2(d_3 - d_1) - e_1(d_3 - d_2) \end{bmatrix} & \begin{bmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ & = \begin{bmatrix} e_2d_3 & e_2d_3 & e_2d_3 \end{bmatrix} \end{aligned}$$

REFERENCES

- [1] M. Alberich-Carramiñana, J. Álvarez Montaner, and F. Dachs-Cadefau, *Multiplier ideals in two-dimensional local rings with rational singularities* (2014), 32 pp., arXiv:1412.3605v1 [math.AG]. $\uparrow 1$
- [2] R. Blanco and S. Encinas, *A procedure for computing the log canonical threshold of a binomial ideal* (2014), 28 pp., arXiv:1405.3942v2 [math.AG]. $\uparrow 7$
- [3] A. Bravo, *A remark on strong factorizing resolutions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **107** (2013), no. 1, 53–60, DOI 10.1007/s13398-012-0080-8. MR3031261 $\uparrow 2$
- [4] M. Blickle, *Multiplier ideals and modules on toric varieties*, Math. Z. **248** (2004), no. 1, 113–121, DOI 10.1007/s00209-004-0655-y. MR2092724 (2006a:14082) $\uparrow 3, 4$
- [5] E. Eisenstein, *Generalizations of the restriction theorem for multiplier ideals* (2010), 17 pp., arXiv:1001.2841v1 [math.AG]. $\uparrow 2$
- [6] C. Galindo and F. Monserrat, *The Poincaré series of multiplier ideals of a simple complete ideal in a local ring of a smooth surface*, Adv. Math. **225** (2010), no. 2, 1046–1068, DOI 10.1016/j.aim.2010.03.008. MR2671187 (2012a:14039) $\uparrow 1$
- [7] P. D. González Pérez and B. Teissier, *Embedded resolutions of non necessarily normal affine toric varieties*, C. R. Math. Acad. Sci. Paris **334** (2002), no. 5, 379–382, DOI 10.1016/S1631-073X(02)02273-2 (English, with English and French summaries). MR1892938 (2003b:14019) $\uparrow 6$
- [8] J. A. Howald, *Multiplier ideals of monomial ideals*, Trans. Amer. Math. Soc. **353** (2001), no. 7, 2665–2671 (electronic), DOI 10.1090/S0002-9947-01-02720-9. MR1828466 (2002b:14061) $\uparrow 4$

- [9] ———, *Multiplier Ideals of Sufficiently General Polynomials* (2003), 9 pp., arXiv:math/0303203v1 [math.AG]. ↑5
- [10] E. Hyry and T. Järvilehto, *Jumping numbers and ordered tree structures on the dual graph*, Manuscripta Math. **136** (2011), no. 3-4, 411–437, DOI 10.1007/s00229-011-0449-6. MR2844818 ↑1
- [11] D. Naie, *Jumping numbers of a unibranch curve on a smooth surface*, Manuscripta Math. **128** (2009), no. 1, 33–49, DOI 10.1007/s00229-008-0223-6. MR2470185 (2009j:14034) ↑1
- [12] ———, *Mixed multiplier ideals and the irregularity of abelian coverings of smooth projective surfaces*, Expo. Math. **31** (2013), no. 1, 40–72, DOI 10.1016/j.exmath.2012.08.005. MR3035120 ↑1
- [13] T. Shibuta and S. Takagi, *Log canonical thresholds of binomial ideals*, Manuscripta Math. **130** (2009), no. 1, 45–61, DOI 10.1007/s00229-009-0270-7. MR2533766 ↑5, 8
- [14] K. E. Smith and H. M. Thompson, *Irrelevant exceptional divisors for curves on a smooth surface*, Algebra, geometry and their interactions, Contemp. Math., vol. 448, Amer. Math. Soc., Providence, RI, 2007, pp. 245–254, DOI 10.1090/conm/448/08669, (to appear in print). MR2389246 (2009c:14004) ↑1
- [15] Z. Teitler, *Software for multiplier ideals* (2013), 7 pp., arXiv:1305.4435v1 [math.AG]. ↑7
- [16] H. M. Thompson, *Comments on toric varieties* (2003), 6 pp., arXiv:0310336 [math.AG]. ↑4
- [17] ———, *Multiplier ideals of monomial space curves*, Proc. Amer. Math. Soc. Ser. B **1** (2014), 33–41, DOI 10.1090/S2330-1511-2014-00001-8. MR3168880 ↑1, 2, 5, 6, 7
- [18] K. Tucker, *Jumping numbers on algebraic surfaces with rational singularities*, Trans. Amer. Math. Soc. **362** (2010), no. 6, 3223–3241, DOI 10.1090/S0002-9947-09-04956-3. MR2592954 (2011c:14106) ↑1

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